Lecture 1

Tuesday, June 13, 2017 10:50 PM

Sample Space : $\mathcal{S}$

all possible outcomes of an experiment

Ex: 1. Flip of a coin

$$\mathcal{S} = \{H, T\}$$

2. Roll of a die

$$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$$

3. 3 independent flips of a coin (3 independent trials)

$$\mathcal{S} = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

If we disregard the ordering and just count the number of H's from the 3 flips,

$$\mathcal{S} = \{0, 1, 2, 3\}$$

Lecture 2

Wednesday, June 14, 2017 1:12 PM

Event : collection of outcomes / subset of S.

Eg. 3 (contd.)

Event A: H in the 1^{st} trial

$$A = \{HHH, HHT, HTH, HTT\} \subset S$$

Event B: T in the 3^{rd} trial

$$B = \{HHT, HTT, THT, TTT\} \subset S$$

Impossible Events : ϕ : Empty set

$A \cap B :=$ Intersection of events A & B ("Both A and B happen")

$A \cap B$ = H in 1^{st} trial & T in 3^{rd} trial

$$= \{HHT, HTT\}$$

$A \cup B :=$ Union of events A & B ("Either A or B happens")

$A \cup B$ = H in 1^{st} trial or T in 3^{rd} Trial

$$= \{HHH, \underline{HHT}, HTH, \underline{HTT}, THT, TTT\}$$

$$\boxed{A \cap B \subset A \cup B}$$

A' := Complement of A

("Not A / A does not happen)

$A' = T$ in the 1^{st} trial

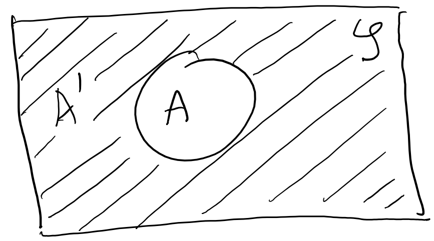
$= \{THH, THT, TTH, TTT\}$

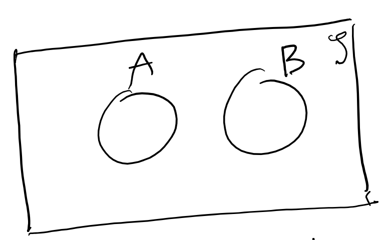
$$\boxed{\begin{array}{l} A \cap A' = \phi \\ A \cup A' = S \end{array}} \longrightarrow A + A' = S$$

A and B have no outcomes in common, i.e. $A \cap B = \phi$: Mutually Exclusive/Disjoint events A & B

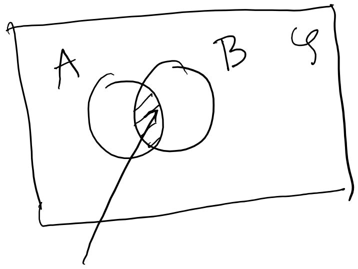
$A \cup B = A + B$

Venn Diagrams:

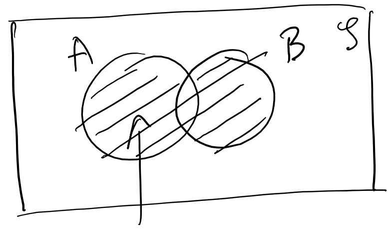




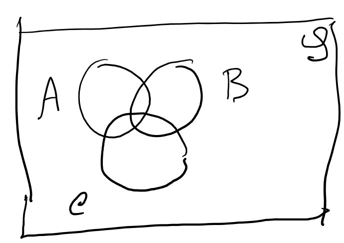
$A \cap B = \phi$



$A \cap B$



$A \cup B$



Sec 2.2: $\underbrace{\text{Probability (of A)}}_{P(A)\,/\,p(A)\,/\,Pr(A)}$: The chance that A will occur

For any event A, $0 \le P(A) \le 1$.

$\uparrow$ Impossible event

$\uparrow$ Certain event

Axioms of Prob:

(1) For any event A, $P(A) \geq 0$.

(2) $P(S) = 1$ [e.g. 1. $P(H \text{ or } T) = 1$]

(3) If $A_1, A_2, \ldots$ (finite / countably infinite)

such that $A_i \cap A_j = \phi$

then $P(A_1 \cup \ldots \cup A_K \cup \ldots)$

$= P(A_1) + \ldots + P(A_K) + \ldots$

i.e. $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

i) Let $A_1 = A_2 = A_3 = \ldots = \phi$

$A_i \cap A_j = \phi$

From A3, $P(\phi) = \sum_{i=1}^{\infty} P(\phi) \Rightarrow \boxed{P(\phi) = 0}$

ii) $A \cap A' = \phi$

By A3, $P(A \cup A') = P(A) + P(A')$ (for A & A')

$\overset{\parallel}{P(S)} = 1$ (By A2)

$$\boxed{P(A) + P(A') = 1}$$

$$\boxed{P(A) + P(A') = 1}$$

Moreover, since $P(A') \geq 0$, $P(A) = 1 - P(A') \leq 1$

Whenever, $A_1 \cup A_2 \cup \dots \cup A_K = \mathcal{S}$

We call the collection <u>exhaustive</u>.

Lecture 3

Thursday, June 15, 2017 1:00 PM

Book → Disjoint events : $A_i \cap A_j = \phi \quad \forall i, j \ (i \neq j)$

Not in book:
- Mutually exclusive: $\bigcap_{i=1}^{K} A_i = \phi$
- Pairwise disjoint/exclusive: $A_i \cap A_j = \phi \quad \forall i \neq j$ $(i, j = 1, \ldots, K)$

Collectively Exhaustive: $\bigcup_{i=1}^{K} A_i = S$

$A \cup B$: at least one of A and B occurs.

(iii) By A3, $A_1, A_2, \ldots$ disjoint events.

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

Finite case → $A_1, \ldots, A_K$ are disjoint

Take $A_{K+1} = A_{K+2} = \ldots = \phi$

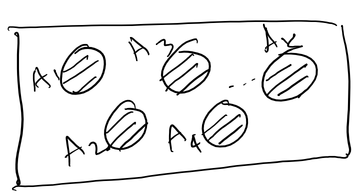
Now, $A_1, A_2, \ldots$ disjoint events (countably infinite)

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots \quad (\text{By } A_3)$$

$$\downarrow \qquad\qquad \downarrow$$

$$P(A_1 \cup \ldots \cup A_K) = P(A_1) + \ldots + P(A_K)$$

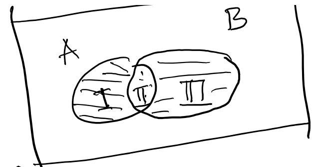
Countable additivity $\Rightarrow$ Finite additivity



(iv) A, B are not disjoint (i.e. $A \cap B \neq \phi$)

$$P(A \cup B) = P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})$$

$$P(A \cup B) = P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})$$



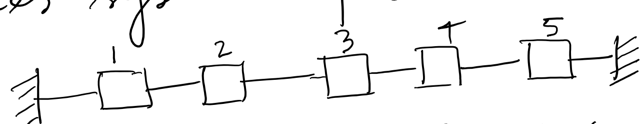
$$= P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) \quad (\text{By } A3)$$

$$= [P(\mathrm{I}) + P(\mathrm{II})] + [P(\mathrm{II}) + P(\mathrm{III})] - P(\mathrm{II})$$

$$= P(A) + P(B) - P(A \cap B)$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

Ex: A series system of 5 identical components



The system works if all components work.

For each component, $P(S) = 0.9 = 1 - P(F)$

$P(\text{System Failure}) = ?$

System Failure $= \{FSSSS, SFSSS, \ldots, SSSSF, FFSSS, \ldots, FFFFF\}$: 31 outcomes

$P(\text{System Success}) = P(SSSSS)$

$= (0.9)^5 \simeq 0.59$

$P(\text{System Failure}) = 1 - 0.59 = 0.41$

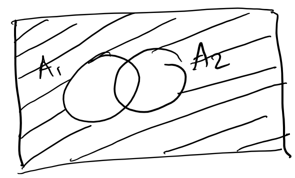
Ex: A firm bids on 3 projects

$A_i = \{\text{Project } i \text{ is awarded}\}$

$P(A_1) = .22, \; P(A_2) = .25, \; P(A_3) = .28$

$$P(A_1 \cap A_2) = 0.11, \quad P(A_1 \cap A_3) = 0.05$$
$$P(A_2 \cap A_3) = 0.07$$
$$P(A_1 \cap A_2 \cap A_3) = 0.01$$

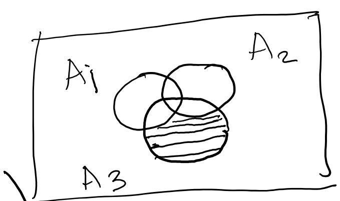
- $P(A_1' \cap A_2') = P\left[(A_1 \cup A_2)'\right]$
$$= 1 - P(A_1 \cup A_2)$$
$$= 1 - \left[P(A_1) + P(A_2) - P(A_1 \cap A_2)\right]$$
$$= 0.64 \text{ (Check)}$$



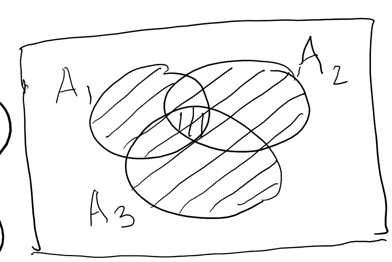
$(A_1 \cup A_2)'$

$$\boxed{\begin{aligned}&(A \cup B)' \\ &= A' \cap B'\end{aligned}}$$

- $A_1' \cap A_2' \cap A_3 = P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$
$$= 0.17 \text{ (Check)}$$



- $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$
$$- P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$
$$= 0.53 \text{ (Check)}$$



$$\boxed{\begin{aligned}&P(A \cup B) \\ &= P(A) + P(B) \\ &\quad - P(A \cap B)\end{aligned}}$$

Inclusion
-Exclusion
Principle

Lecture 4

Friday, June 16, 2017 1:17 PM

Flip of a coin : $\mathcal{S} = \{H, T\}$

A1. $P(H), P(T) \geq 0$

A2. $P(\mathcal{S}) = P(H) + P(T) = 1$

$P(H) = P(T) = 0.5$: Fair Coin

or, $P(H) = 0.7 = 1 - P(T)$

<u>Frequentist</u> : Perform the experiment indep. ly and identically over a long run.

Probability $\approx$ Relative Freq. / Proportion

<u>Subjective / Personal Prob.</u> : eg. Weather

Prior belief $\longrightarrow$ <u>Bayesian</u> Statistics

Let $E_1, E_2, E_3, \ldots$ disjoint, exhaustive simple events

For any event A, let $B_A = \{j : E_j \subset A\}$ be the set of indices of the simple events that make up A.

$$\bigcup_{j \in B_A} E_j = A$$

$$P(A) = \sum_{j \in B_A} P(E_j)$$

Let $\mathcal{S} = E_1 \cup E_2 \cup \ldots \cup E_N$: Disjoint, i.e. $E_i \cap E_j = \phi \ \forall i \neq j$ $(i, j = 1, 2, \ldots, N)$

$P(E_1) = \ldots = P(E_N)$: Equally likely

$$1 = P(\mathcal{S}) = \sum_{j=1}^{N} P(E_j) \qquad (B_{\mathcal{S}} = \{1, \ldots, N\})$$

$$= N \cdot P(E_1)$$

$$\Rightarrow P(E_1) = \ldots = P(E_N) = \frac{1}{N}$$

For any A, $P(A) = \sum_{j \in B_A} \frac{1}{N} = \frac{N(A)}{N}$ $\left[N_A = |B_A| = \# E_j\text{'s making up } A \right]$

Eg. 1. Flip of a fair coin: $(N = 2)$

$$E_1 = \{H\}, \ E_2 = \{T\}$$

$$\mathcal{S} = E_1 \cup E_2 = \{H, T\}$$

$$\underbrace{P(E_1)}_{P(H)} = \underbrace{P(E_2)}_{P(T)} = \frac{1}{2}$$

2. Roll of a fair die: $(N=6)$

$$E_j = \{j\} \quad \forall j = 1, 2, \ldots, 6$$

$$S = \bigcup_{j=1}^{6} E_j = \{1, 2, \ldots, 6\}$$

$$P(E_j) = P(\{j\}) = \frac{1}{6} \quad \forall j = 1, \ldots, 6$$

Now, $P(\text{Value} \le 3) = P(\{1, 2, 3\})$

$$= P(E_1 \cup E_2 \cup E_3)$$

$$= 3/6 = 0.5$$

Counting Techniques:

① Product Rule: In an ordered pair, 1^{st} element can be selected in n_1 ways and for each of those ways, the 2^{nd} element can be selected in n_2 ways, then there are $n_1 n_2$ possible pairs.

It can be easily extended for ordered K-tuples. $(K>2)$

② Permutation:

(i) Ordered K-tuple with elements taken from K distinct objects. Same Object cannot appear more than once.

Q: How many K-tuples can we have?

$$\left.\begin{array}{lllll} \text{Places:} & 1^{st} & 2^{nd} & \ldots & K^{th} \\ \text{\# Ways:} & K & (K-1) & \ldots & 1 \end{array}\right\}$$ Permutation of K distinct objects (order matters):

$$P_K^K := K(K-1)\ldots 3\cdot 2\cdot 1 = K!$$

(ii) Ordered K-tuple with elements taken from $m\,(>K)$ distinct objects. Q: How many K-tuples?

$$\left.\begin{array}{lllll} \text{Places:} & 1^{st} & 2^{nd} & \ldots & K^{th} \\ \text{\# Ways:} & m & m-1 & \ldots & m-K+1 \end{array}\right\} \; P_m^K = P_{K,m} :=$$

$$m(m-1)\ldots(m-K+1)$$

$$= \frac{m(m-1)\ldots 3\cdot 2\cdot 1}{(m-K)(m-K-1)\ldots 3\cdot 2\cdot 1}$$

$$= \frac{m!}{(m-K)!}$$

③ Combination: Unordered Collection

(i) K distinct Objects $\Rightarrow$ 1 combination of size K

($K!$ permutations)

(iii) $m\ (>K)$ distinct objects $\Rightarrow \binom{m}{K}$: # of combinations of size K from a set of m elements

$$\binom{m}{K} = \frac{P_m^K}{K!}$$

$P_m^K \longrightarrow$ Total # of ordered K-tuples

$K! \longrightarrow$ # of ordered K-tuples for each combination

$$= \frac{m!}{(m-K)!\,K!}$$

Toss a coin twice: (Fair Coin)

$\mathcal{S} = \{HH, HT, TH, TT\}$

Simple events: $E_1 = \{HH\}$
$E_2 = \{HT\}$
$E_3 = \{TH\}$
$E_4 = \{TT\}$

$$P(\{HH\}) = P(\{HT\}) = P(\{TH\}) = P(\{TT\}) = \frac{1}{4}$$

We'll often write: $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$

Eg. 4 elements: A, B, C, D $(m=4)$

$(K=3)$

Permutations: $P^3_4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 24$

$3! = 6$

A, B, C	A, B, D	A, C, D	B, C, D
A, C, B	A, D, B	A, D, C	B, D, C
B, A, C	B, A, D	C, A, D	D, B, C
B, C, A	B, D, A	C, D, A	D, C, B
C, A, B	D, A, B	D, A, C	C, B, D
C, B, A	D, B, A	D, C, A	C, D, B

Combinations: $\{ABC\}, \{ABD\}, \{ACD\}, \{B,C,D\}$

$$P^3_4 = \binom{4}{3} \times 3!$$

[# Permutations = # Combinations × # ways each combination can be ordered]

$$\binom{4}{3} = \frac{P^3_4}{3!} = \frac{24}{6} = 4$$

In general, $\binom{m}{K} = \frac{P^K_m}{K!} = \frac{m!}{(m-K)!\,K!}$

Eg. 5 components in a serial connection

$|\{\text{System Succeeds}\}| = |\{SSSSS\}|$

outcomes/simple events in $\{\text{System Fails}\} = \underbrace{32}_{|\mathcal{S}|} - 1 = 31$

Alternatively, $\{\text{System Fails}\} = \{1 \text{ component Fails}\} \cup \{2 \text{ components fail}\} \cup \dots \cup \{5 \text{ components fail}\}$

[This collection is disjoint]

$$|\{1 \text{ component fails}\}| = \binom{5}{1} = \binom{5}{4} = 5$$
$$|\{2 \text{ components fail}\}| = \binom{5}{2} = \binom{5}{3} = 10$$
$$|\{3 \text{ components fail}\}| = \binom{5}{3} = \binom{5}{2} = 10$$
$$|\{4 \text{ components fail}\}| = \binom{5}{4} = \binom{5}{1} = 5$$
$$|\{5 \text{ components fail}\}| = \binom{5}{5} = \binom{5}{0} = 1$$

$\boxed{0! = 1}$

$$\Rightarrow |\{\text{System Fails}\}| = 5 + 10 + 10 + 5 + 1$$

In general, $\binom{K}{0} = \binom{K}{K} = 1$, $\binom{m}{K} = \binom{m}{m-K} = \frac{m!}{K!\,(m-K)!}$

and $\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{m} = 2^m$

Here, $\binom{K}{K} = \frac{K!}{K!\,0!} = 1$

and $\binom{K}{0} = \frac{K!}{0!\,K!} = 1$

Eg: 25 printers → 10 laser printers, 15 inkjet printers. If 6 printers are selected, What is the prob.

} Think of printers marked with numbers 1, 2, ..., 25

Eg: 25 printers → 10 laser printers, 15 inkjet printers ... } marked with numbers 1, 2, ..., 25

If 6 printers are selected, what is the prob. that exactly 3 are laser printers?

Let $A = \{\text{Exactly 3 laser printers}\} = \{\text{3 laser, 3 inkjet printers}\}$

All possible printer selections are equally likely.

$$\therefore \; P(A) = \frac{N(A)}{N}$$

ways to choose 3 laser printers out of 10 such printers

$$= \frac{\binom{10}{3} \times \binom{15}{3}}{\binom{25}{6}}$$

ways to choose 3 inkjet printers out of 15 such printers

Product Rule

ways to choose 6 printers out of 25 printers

$$= \frac{\dfrac{10!}{3!\,7!} \times \dfrac{15!}{3! \times 12!}}{\dfrac{25!}{6!\,19!}} = \frac{120 \times 455}{177100} = 0.3083$$

(Contd.)

$P(\text{at least 3 laser printers})$
$= P(\text{exactly 3 l.p.'s}) + P(\text{exactly 4 l.p.'s}) + P(\text{exactly 5 l.p.'s}) + P(\text{exactly 6 l.p.'s})$

$$= \frac{\binom{10}{3}\times\binom{15}{3} + \binom{10}{4}\times\binom{15}{2} + \binom{10}{5}\times\binom{15}{1} + \binom{10}{6}\times\binom{15}{0}}{\binom{25}{6}}$$

Finish ...

Permutation: $P(\text{exactly 3 l.p.'s}) = \dfrac{\binom{10}{3}\times\binom{15}{3}\times 6!}{P^6_{25}} = \dfrac{\binom{10}{3}\times\binom{15}{3}}{\frac{25!}{19!\,6!}}$

$$= \frac{\binom{10}{3}\times\binom{15}{3}}{\binom{25}{6}}$$

Ex. (2.42) 3 couples, 6 seats in a row in a theater and they randomly choose seats. One couple is: Jim (J) & Paula (P).

(i) What's the Prob. that J & P sit on the far left?

Combination: $\dfrac{1}{\binom{6}{2}} = \dfrac{1}{15}$ $\left[\binom{n}{2} = \dfrac{n!}{2!\,(n-2)!} = \dfrac{n(n-1)}{2}\right]$

Permutation: $\dfrac{2}{P^2_6} = \dfrac{2}{6!/4!} = \dfrac{2}{30} = \dfrac{1}{15}$

alternatively, $\dfrac{2\times 4!}{6!} = \dfrac{1}{15}$

(ii) What's the Prob. that J & P sit next to each other?

Combination: $\dfrac{5}{\binom{6}{2}}$ $\left|\{(1,2), (2,3), (3,4), (4,5), (5,6)\}\right|$

$= \dfrac{1}{3}$

Switching J & P

Permutation: $\dfrac{5\times 2\times 4!}{6!}$ Arranging all others

Choosing J, P's seats

$= \dfrac{1}{3}$

(iii) What's the Prob. that at least one couple sits together?

1	2	3	4	5	6

O & X H & W (O) H & W (X)

? × ? × 2 = 8

Let's look at the complement

1	2	3	4	5	6
J	O	P	X	O	X
J	O	X	P	O	X
J	O	X	O	P	X
J	O	X	O	X	P

$2 \times 2 \times 2 = 8$

$2 \times 2 \times 2 \times 2 = 16$

H & W (O)   H & W (X)   2 & 3   5 & 6

$2 \times 2 \times 2 = 8$

H & W (O)   H & W (X)   O & X

$2 \times 2 \times 2 = 8$

J & P   3 Couples

$$(8+16+8+8) \times 2 \times 3 = 240$$

$$P(\text{no couple sits together}) = \frac{240}{6!} = \frac{1}{3}$$

$$P(\text{at least one couple sits together}) = 1 - 1/3 = \frac{2}{3}$$